C3-R4: MATHEMATICAL METHODS FOR COMPUTING

NOTE:

1.	**Answer question 1 and any FOUR from questions 2 to 7.**
2.	**Parts of the same question should be answered together and in the same sequence.**

Time: 3 Hours **Total Marks: 100**

1.

a) A new computer virus can enter the system through an e-mail or through the internet. There is a 30% chance of receiving it through an email. There is a 40% chance of receiving it through the internet. Also, there is 15% chance that the virus enters the system both through an e- mail and the internet. What is the probability that the virus does not enter the system at all?

b) Consider two discrete random variables X and Y with joint probability mass function given in the following table:

		Y: -1	Y: 0	Y: 1
X	-2	1/16	1/16	1/16
	-1	1/8	1/16	1/8
	1	1/8	1/16	1/8
	2	1/16	1/16	1/16

Show that X and Y are not independent variables.

c) Let $f(x) = x,\ 0 < x < 2\pi$

Find the Fourier coefficients a_n and b_n, $n = 1, 2, 3, \ldots$, for $f(x)$.

d) Solve the following linear programming problem

Maximize $z = 5x_1 + 4x_2$

subject to

$6x_1 + 4x_2 \leq 24$

$x_1 + 2x_2 \leq 6$

$x_1, x_2 \geq 0$

e) Assume that the number of messages input to a communication channel in an interval of duration t seconds is Poisson distributed with parameter 0.3t. Compute the probability that exactly three messages arrive during a 10 second interval.

f) Find the Laplace transform of the function $f(t) = e^{-t} \cos 2t$.

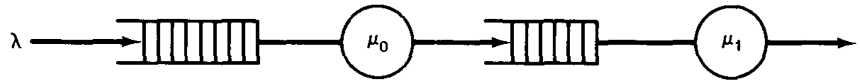
g) With usual notations, prove that $H(X, Y) = H(Y) + H(X|Y)$ for two random variables X and Y which take on respective values $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ with joint probability mass function $p(x_i, y_i) = P\{X = x_i, Y = y_j\}$.

(7x4)

2.

a) At a telephone booth, if the mean duration of telephone conversation is three minutes, and that no more than a three – minute (average) wait for the phone may be tolerated then determine the maximum call rate that can be supported by one telephone booth.

b) A repair facility shared by a large number of machines has two sequential stations with respective rates 1 per hour and 2 per hour.



The cumulative failure rate of all the machines is 0.5 hour. Assume that the system behaviour may be approximated by the two-stage tandem queue in above figure, determine the average repair time.

(9+9)

3.

a) A Markov chain $X_0, X_1, \ldots$ on states 0,1,2 has the transition probability matrix

$P = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.9 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \end{bmatrix}$ and initial distribution

$p_0 = \Pr\{X_0 = 0\} = 0.3, p_1 = \Pr\{X_0 = 1\} = 0.4,$ and $p_2 = \Pr\{X_0 = 2\} = 0.3.$

Determine $\Pr\{X_0 = 0, X_1 = 1, X_2 = 2\}$.

b) Obtain a Fourier series expression for $f(x) = x^3$, $-\pi < x < \pi$.

(10+8)

4.

a) Suppose X and Y are random variables that assume the values x and y, where x = 1 or 2 and y = 1,2,3,4, with probabilities given by the following table:

	1	2	3	4
1	1/4	1/8	1/16	1/16
2	1/16	1/16	1/4	1/8

Obtain the entropy H(X, Y).

b) A pair of fair dice is rolled. Let $X = \begin{cases} 1 \text{ if the sum of the dice is 6} \\ 0 \text{ otherwise} \end{cases}$

and let Y equals the value on the first die. Find the joint probability mass function of (X, Y).

(10+8)

5.

a) Find the inverse Laplace transform of the function F(s) = $\dfrac{s}{s^2 + 4s + 13}$.

b) Using the Laplace transform, find the solution of the initial value problem $y'' + 25y = 10\cos 5t, y(0) = 2, y'(0) = 0$.

(8+10)

6.

a) If the probability density function of a random variable x, $0 \le x \le 1$, is $f(x) = \frac{e^x}{e-1}$, then find the cumulative probability density function F(x) of x. Using it and the inverse transform method, determine the random variable x.

b) Use 10 random numbers between [0, 1] and Monte Carlo simulation to estimate the following integral

$$\int_0^1 (1-x^2)^{3/2} dx$$

Also compare your estimated answer with the exact integral value.

(9+9)

7.

a) Solve the following linear programming problem using the simplex method

Maximize $z = 5x_1 + 3x_2$

subject to

$3x_1 + 5x_2 \le 15$

$5x_1 + 2x_2 \le 10$

$x_1, x_2 \ge 0.$

b) Consider the function for $x_1 \ge 0$, $x_2 \ge 0$, $f(x_1, x_2) = \ell n\,(x_1 + 1) + x_2$, show that f is a concave function. Use this knowledge and the Karush Kuhn Tucker condition to prove that (0, 3) is an optimal solution of the following programming problem:

Minimize $f(x_1, x_2) = \ell n\,(x_1 + 1) + x_2$

subject to

$2x_1 + x_2 \le 3$

$x_1 \ge 0, x_2 \ge 0.$

(9+9)